

ON ORBITS IN DOUBLE FLAG VARIETIES FOR SYMMETRIC PAIRS

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Dedicated to Jiro Sekiguchi on his 60th birthday.

ABSTRACT. Let G be a connected, simply connected semisimple algebraic group over the complex number field, and let K be the fixed point subgroup of an involutive automorphism of G so that (G, K) is a symmetric pair.

We take parabolic subgroups P of G and Q of K respectively and consider the product of partial flag varieties G/P and K/Q with diagonal K -action, which we call a *double flag variety for a symmetric pair*. It is said to be of *finite type* if there are only finitely many K -orbits on it.

In this paper, we give a parameterization of K -orbits on $G/P \times K/Q$ in terms of quotient spaces of unipotent groups without assuming the finiteness of orbits. As a result, we get several useful criteria for the finiteness of orbits. If one of $P \subset G$ or $Q \subset K$ is a Borel subgroup, the finiteness of orbits is closely related to spherical actions. In such cases, the criteria enable us to obtain a complete classification of double flag varieties of finite type. As a consequence, we obtain classifications of K -spherical flag varieties G/P and G -spherical homogeneous spaces G/Q .

INTRODUCTION

Let G be a connected, simply connected semisimple algebraic group over the complex number field \mathbb{C} .

Let P_i ($i = 1, 2, \dots, k$) be parabolic subgroups of G and consider partial flag varieties $\mathfrak{X}_{P_i} := G/P_i$ of G . We are interested in the product of flag varieties $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \cdots \times \mathfrak{X}_{P_k}$, on which G acts diagonally. We say a multiple flag varieties $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \cdots \times \mathfrak{X}_{P_k}$ is of *finite type* if it admits only finitely many G -orbits. It is an interesting problem to classify multiple flag varieties of finite type. According to a result of Magyar-Weyman-Zelevinsky ([MWZ99, MWZ00]), k must be less than or equal to 3 if a multiple flag variety is of finite type and if G is of classical type.

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Let us consider a triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$. If $P_3 = B$ is a Borel subgroup, then the triple flag variety is of finite type if and only if $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2}$ is a spherical G -variety. For maximal parabolic subgroups P_1 and P_2 , Littelmann classified such spherical double flag varieties ([Lit94]). For general parabolic subgroups P_1 and P_2 , Stembridge [Ste03] classified them completely. In [MWZ99, MWZ00], a classification of G -orbits on the triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$ was presented for type A and C if it is of finite type.

Let K be a subgroup of fixed points of a non-trivial involution θ of G . Take parabolic subgroups P of G and Q of K . Then we call $\mathfrak{X}_P \times \mathcal{Z}_Q (= G/P \times K/Q)$ a *double flag variety for a symmetric pair* (G, K) , where $\mathcal{Z}_Q := K/Q$ (see [NO11]). The subgroup K naturally acts on $\mathfrak{X}_P \times \mathcal{Z}_Q$ diagonally. This notion is a generalization of the triple flag varieties (see § 6).

In some cases, finiteness of G -orbits on a triple flag variety implies finiteness of K -orbits on a double flag variety for (G, K) . In [NO11], we investigated such situations and got two sufficient conditions for $\mathfrak{X}_P \times \mathcal{Z}_Q$ to be of finite type. The first one is

Theorem 1 ([NO11, Theorem 3.1]). *Let P' be a θ -stable parabolic subgroup of G such that $P' \cap K = Q$. If the number of G -orbits on $\mathfrak{X}_P \times \mathfrak{X}_{\theta(P)} \times \mathfrak{X}_{P'}$ is finite, then there are only finitely many K -orbits on the double flag variety $\mathfrak{X}_P \times \mathcal{Z}_Q$.*

Here is the second one.

Theorem 2 ([NO11, Theorem 3.4]). *Let P_i ($i = 1, 2, 3$) be a parabolic subgroup of G . Suppose that $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$ has finitely many G -orbits and that $Q := P_1 \cap P_2$ is a parabolic subgroup of K . Then $\mathfrak{X}_{P_3} \times \mathcal{Z}_Q$ has finitely many K -orbits.*

Moreover, if P_3 is a Borel subgroup B and the product $P_1 P_2$ is open in G , then the converse is also true, i.e., the double flag variety $\mathfrak{X}_B \times \mathcal{Z}_Q$ is of finite type if and only if the triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_B$ is of finite type.

Using these two theorems, we can produce many examples of double flag varieties of finite type. However, a complete classification is not known yet.

In this paper, we study K -orbit structure on a general double flag variety $\mathfrak{X}_P \times \mathcal{Z}_Q$ which is not necessarily of finite type and, as a result, we get some efficient criteria for the finiteness of orbits. Our method relies on the Bruhat decomposition and “KGB-decomposition” (i.e., the K -orbit decomposition of flag varieties; see § 2.2). The set of K -orbits in $\mathfrak{X}_P \times \mathcal{Z}_Q$ is decomposed into a finite disjoint union of some quotient spaces parameterized by elements of Weyl groups (or “Bruhat parameters”) and “KGB-parameters”. For each of these parameters, we construct a certain double coset space of unipotent subgroups related to P and Q , which admits an action of a subgroup of Levi component of Q . The quotient spaces are obtained from this action.

Though general description of the orbit space structure of $K \backslash (\mathfrak{X}_P \times \mathcal{Z}_Q)$ is much complicated, it becomes considerably simple if Q is a Borel subgroup of K . So let us give a parameterization of orbits in this special case here, and for a general case we refer to Theorem 2.7, which is the first main result of the paper.

Let B be a θ -stable Borel subgroup of G which contains a θ -stable maximal torus T , and W the Weyl group of G . We denote by U_B the unipotent radical of B so that $B = TU_B$. We write $B_K = B \cap K = T_K U_{B_K}$, a Borel subgroup of K .

Theorem 3. *Let P be a standard parabolic subgroup of G containing B , and W_P a subgroup of W corresponding to the Levi component of P . Then the K -orbits on the double flag variety are parameterized as follows:*

$$K \backslash (\mathfrak{X}_P \times \mathcal{Z}_{B_K}) \simeq \coprod_{w \in W_P \backslash W} \left((w^{-1} P w \cap U_B) \backslash U_B / U_{B_K} \right) / T_K,$$

where the maximal torus T_K of K acts on the double coset space via conjugation.

Let us discuss finiteness of K -orbits on the double flag varieties. If P or Q is a Borel subgroup of G or K respectively, we can do detailed analysis.

If $Q = B_K$, Theorem 3 can be applied to get a criterion for $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ to be of finite type (Corollary 4.3). In this case, a double flag variety $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ is of finite type if and only if \mathfrak{X}_P is K -spherical. Moreover, if we denote by \mathcal{O} the closed K -orbit through the base point $e \cdot P$ in \mathfrak{X}_P (recall that we assume P contains the θ -stable Borel subgroup B), it is equivalent to say that the conormal bundle $T_{\mathcal{O}}^* \mathfrak{X}_P$ (or the normal bundle $T_{\mathcal{O}} \mathfrak{X}_P$) is spherical. In this respect, we can apply Panyushev's theorem to obtain that the conormal bundle $T_{\mathcal{O}'}^* \mathfrak{X}_P$ (or the normal bundle $T_{\mathcal{O}'} \mathfrak{X}_P$) is K -spherical for any K -orbit \mathcal{O}' if the flag variety \mathfrak{X}_P is K -spherical (see [Pan99]). We refer the details to Theorem 4.5. As a consequence, we see that $\mathfrak{X}_P \times \mathcal{Z}_Q$ is of finite type if and only if the action of Levi component of $P \cap K$ on the fiber of $T_{\mathcal{O}} \mathfrak{X}_P$ at $e \cdot P$ is a multiplicity free action. Using the tables of multiplicity free actions by Benson-Ratcliff [BR96], we obtain a complete classification of the double flag varieties $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ of finite type in Theorem 7.3 in § 7:

Theorem 4. *Let G be a connected simple algebraic group. Then a double flag variety $G/P \times K/B_K$ is of finite type if and only if it appears in Table 2. The table also serves as a complete list of K -spherical partial flag varieties G/P .*

On the other hand, if $P = B$, a double flag variety $\mathfrak{X}_B \times \mathcal{Z}_Q$ is of finite type if and only if G/Q is a G -spherical variety. In this case, Theorem 2.7 implies that a double flag variety is of finite type if and only if a certain linear action of a reductive subgroup of K is a multiplicity free action (Theorem 5.2). So we can again use tables in [BR96] to get a classification of such double flag varieties of finite type. See Theorem 8.3 and Table 3. We note that if $\mathfrak{X}_B \times \mathcal{Z}_Q$ is of finite type, then $\mathfrak{X}_{P'} \times \mathcal{Z}_{B_K}$ is also of finite type, where P' is a θ -stable parabolic subgroup of G such that $P' \cap K = Q$.

Another motivation to study double flag varieties $\mathfrak{X}_B \times \mathcal{Z}_Q$ of finite type comes from the theory of character sheaves. Character sheaves were first introduced by Lusztig [Lus]. There are certain G -equivariant simple perverse sheaves on G , which provide a geometric theory of characters of a connected reductive group over an arbitrary algebraically closed field.

Recently, some generalizations of character sheaves have been studied. Finkelberg, Ginzburg and Travkin developed the theory of mirabolic character sheaves in [FG10] and [FGT09]. Following the work of Kato [Kat09], Henderson and Trapa suggested the theory of exotic character sheaves in [HT11]. These character sheaves are certain K -equivariant simple perverse sheaves on $V \times G/K$, where V is some K -module. Here in the mirabolic case, $(G, K, V) = (GL_n \times GL_n, (GL_n)_{\text{diag}}, \mathbb{C}^n)$ and in the exotic case, $(G, K, V) = (GL_{2n}, Sp_n, \mathbb{C}^{2n})$. A key ingredient is that there are only finitely many K -orbits on the generalized flag $V \times G/B$.

One may hope that there is a generalization of character sheaves on $K/Q \times G/K$, which generalizes both the mirabolic character sheaves and exotic character sheaves. In order to do this, one first need to know when a double flag variety $\mathfrak{X}_B \times \mathcal{Z}_Q$ has only finitely many K -orbits. We believe that the classification of double flag varieties $\mathfrak{X}_B \times \mathcal{Z}_Q$ of finite type is a necessary ingredient for establishing the (conjectural) generalization of character sheaf theory.

The Robinson-Schensted correspondence is a bijection correspondence between permutations and pairs of standard Young tableaux of the same shape. Steinberg gave a geometric interpretation of this correspondence, by showing that both sides naturally parametrize the irreducible components of the Steinberg variety, which is by definition the conormal variety of the product of flag varieties. It is interesting to study a similar question for conormal variety of double flag of finite type. The mirabolic case was obtained by Travkin [Tra09], Finkelberg-Ginzburg-Travkin [FGT09] and the exotic case was obtained by Henderson-Trapa [HT11], in which they also made some conjectures relating the exotic Robinson-Schensted correspondence to exotic character sheaves.

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1. PRELIMINARIES

1.1. Let G be a connected, simply connected semisimple algebraic group over the complex number field \mathbb{C} and θ a non-trivial involutive automorphism of G . We put $K = G^\theta = \{g \in G : \theta(g) = g\}$, the subgroup of fixed elements of θ , which is connected and reductive by our assumption on G (see [Ste68, Theorem 8.1]). We denote the Lie algebra of G (respectively K) by \mathfrak{g} (respectively \mathfrak{k}). In the following, we use similar notation; for an algebraic group we use a Roman capital letter, and for its Lie algebra the corresponding German small letter.

Let $B \subset G$ be a θ -stable Borel subgroup and take a θ -stable maximal torus T in B . We consider the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$, the Weyl group $W = W_G = N_G(T)/Z_G(T)$ with respect to T , and the positive system Δ^+ corresponding to B . Then Δ^+ determines a simple system Π . Since B and T are θ -stable, θ naturally acts on W_G and Δ , and preserves Δ^+ and Π .

We say that a parabolic subgroup P of G is standard if $P \supset B$. There exists a one-to-one correspondence between the standard parabolic subgroups P and the subsets $J \subset \Pi$; the root subsystem Δ_J generated by J is the root system of the standard Levi component L of P . Notice that the θ -stable parabolic subgroups correspond exactly to the θ -stable subsets in Π . If P corresponds to J , then we will write $P = P_J$ and write $P_J = L_J U_J$ for the Levi decomposition, where L_J is the standard Levi factor and U_J is the unipotent radical. We denote the Weyl group of Δ_J by W_J and put $W^J := W/W_J$. For two subsets $J, J' \subset \Pi$, put ${}^J W^{J'} := W_J \backslash W / W_{J'}$. In the following, we often take representatives of elements of W in $N_G(T)$ and regard W as a subset of $N_G(T)$ or of G . Similarly, we take representatives of elements of $W^J, {}^J W^{J'}$ in W and then in $N_G(T)$ so that we have $W^J, {}^J W^{J'} \subset N_G(T)$.

1.2. In this subsection, we assume that G is a connected reductive algebraic group over \mathbb{C} . Let X be an irreducible normal G -variety. If X has an open B -orbit for a Borel subgroup B of G , it is called a spherical variety and the G -action is called a spherical action. It is well-known that X is spherical if and only if there are only finitely many B -orbits in X ([Bri86, Vin86]). The G -action on X induces a G -action on the ring of regular functions $\mathbb{C}[X]$. It is easy to see that if X is spherical, then the decomposition of $\mathbb{C}[X]$ into irreducible G -modules is multiplicity-free. The converse is also true if X is in an affine variety ([VK78]). If X is isomorphic to a vector space and the G -action is linear, $\mathbb{C}[X]$ can be identified with the symmetric algebra $S(X^*)$ of the dual space X^* . We therefore have:

Lemma 1.1. *Let X be a finite-dimensional representation of G . The following conditions are equivalent.*

- (1) *X is a spherical G -variety.*
- (2) *The dual space X^* is a spherical G -variety.*
- (3) *The decomposition of the symmetric algebra $S(X)$ into irreducible G -modules is multiplicity-free.*
- (4) *The decomposition of the symmetric algebra $S(X^*)$ into irreducible G -modules is multiplicity-free.*

Proof. The equivalences (1) \Leftrightarrow (4) and (2) \Leftrightarrow (3) follow from [VK78]. We see

$$\begin{aligned} \mathrm{Hom}_G(V, S(X)) &= \bigoplus_{n=0}^{\infty} \mathrm{Hom}_G(V, S^n(X)) \\ &\simeq \bigoplus_{n=0}^{\infty} \mathrm{Hom}_G(V^*, S^n(X^*)) = \mathrm{Hom}_G(V^*, S(X^*)). \end{aligned}$$

for every irreducible G -module V , where $S^n(X)$ denotes the n -th symmetric tensor of X . Hence we get the equivalence (3) \Leftrightarrow (4). \square

2. DOUBLE FLAG VARIETIES FOR SYMMETRIC PAIRS

Suppose that G is a connected, simply connected semisimple algebraic group over \mathbb{C} with an involutive automorphism θ and let $K := G^\theta$. Let P be a parabolic subgroup of G , and Q a parabolic subgroup of K . We denote the partial flag varieties G/P , K/Q by \mathfrak{X}_P , \mathfrak{Z}_Q , respectively. The product $\mathfrak{X}_P \times \mathfrak{Z}_Q$ is called a *double flag variety for the symmetric pair* (G, K) . We say that a double flag variety $\mathfrak{X}_P \times \mathfrak{Z}_Q$ is of *finite type* if there are only finitely many orbits on the product $\mathfrak{X}_P \times \mathfrak{Z}_Q$ with respect to the diagonal K -action (see [NO11]).

In this article, we study the structure of the orbit space $K \backslash (\mathfrak{X}_P \times \mathfrak{Z}_Q)$, and give a parameterization of orbits. As a consequence of the parameterization, we get a criterion to determine if the double flag variety is of finite type.

It is known that there exists a θ -stable parabolic subgroup P' of G such that $Q = P' \cap K$ ([BH00, Theorem 2]). Then by replacing P with its conjugate subgroup, we may assume that P and P' are standard parabolic subgroups for a θ -stable Borel subgroup B . We use notations in § 1.1 for our G , K , and B . Write $J, J' \subset \Pi$ for the subsets such that $P = P_J$ and $P' = P_{J'}$.

We parameterize K -orbits on $\mathfrak{X}_P \times \mathfrak{Z}_Q$ using reduction by two well-known decompositions: the Bruhat decomposition and the KGB decomposition.

First we reduce the orbit space by the Bruhat decomposition.

2.1. Reduction to Bruhat decomposition. Notice that there is a bijection

$$K \backslash (\mathfrak{X}_P \times \mathfrak{Z}_Q) \simeq P \backslash G / Q, \quad K \cdot (gP, kQ) \mapsto Pg^{-1}kQ \quad (g \in G, k \in K). \quad (2.1)$$

Since $Q = K \cap P'$, we have the following reduction map Φ .

$$\begin{array}{ccc} K \backslash (\mathfrak{X}_P \times \mathfrak{Z}_Q) & \xrightarrow{\sim} & P \backslash G / Q \\ & \searrow \Phi & \downarrow \text{proj} \\ & & P \backslash G / P' = \coprod_{w \in {}^J W^{J'}} PwP' \\ & & \downarrow \simeq \\ & & {}^J W^{J'} = W_J \backslash W / W_{J'} \end{array}$$

Thus the determination of the orbit structure $K \backslash (\mathfrak{X}_P \times \mathfrak{Z}_Q)$ reduces to the analysis of the fiber

$$\Phi^{-1}(w) \simeq P \backslash PwP' / Q \quad (w \in {}^J W^{J'}). \quad (2.2)$$

Let us fix $w \in {}^J W^{J'}$ in the following. We put

$$P^w := w^{-1}Pw \quad \text{and} \quad P^w \cap P' := w^{-1}Pw \cap P'. \quad (2.3)$$

More generally, we write $H^g = g^{-1}Hg$ for any subgroup $H \subset G$ and $g \in G$.

Lemma 2.1. *The map $(P^w \cap P') \backslash P' / Q \rightarrow P \backslash PwP' / Q$ given by $(P^w \cap P') a Q \mapsto PwaQ$ for $a \in P'$ is bijective.*

This is a consequence of a more general lemma below. In the lemma, G, H, H' refer general groups, which are different from the present notation.

Lemma 2.2. *Let G be a group and H, H' its subgroups. Let $A \subset H'$ be a subset and $g \in G$ an element. Then the map $(H^g \cap H') \backslash (H^g \cap H')A \rightarrow H \backslash HgA$ given by $(H^g \cap H')a \mapsto Hga$ for $a \in A$ is bijective.*

Proof. The surjectivity is clear. If $Hga_1 = Hga_2$ for $a_1, a_2 \in A$, then $a_2a_1^{-1} \in H^g \cap H'$. Hence $(H^g \cap H')a_1 = (H^g \cap H')a_2$ and the map is injective. \square

We apply Lemma 2.2 in the setting where $G = G$, $H = P$, $H' = P'$, $A = P'$, $g = w$. Taking quotients by right Q -action, we get Lemma 2.1.

2.2. Reduction to smaller symmetric spaces. The orbit structure $K \backslash (\mathfrak{X}_P \times \mathfrak{Z}_Q)$ reduces to the structure of fibers (2.2) of the reduction map Φ to the Bruhat decomposition $P \backslash G / P'$. In this subsection, we further reduce it by the KGB decomposition for smaller symmetric spaces.

Put $L'_K := L' \cap K$ and consider $P^w \cap L'$, which is a parabolic subgroup of L' by [Car85, Proposition 2.8.9]. Then L'_K is a symmetric subgroup of L' , and it is known that $(P^w \cap L') \backslash L' / L'_K$ is a finite set. Let us denote this finite set by $\mathcal{V}(w)$ for $w \in {}^J W^{J'}$.

Lemma 2.3. *The map*

$$\Psi_w : (P^w \cap P') \backslash P' / Q \rightarrow (P^w \cap L') \backslash L' / L'_K = \mathcal{V}(w)$$

given by $(P^w \cap P') a Q \mapsto (P^w \cap L') b L'_K$ is well-defined, where $a = bu \in L'U' = P'$ is the Levi decomposition.

Proof. We put $U'_K := U' \cap K$, and let us consider the following diagram.

$$\begin{array}{ccc} (P^w \cap P') \backslash P' / Q & \xrightarrow{=} & (P^w \cap P') \backslash L'U' / L'_K U'_K \\ & \searrow \Psi_w & \downarrow \text{proj.} \\ & & (P^w \cap P') \backslash L'U' / L'_K U' \\ & & \downarrow \wr \downarrow \iota^{-1} \\ & & (P^w \cap L') \backslash L' / L'_K = \mathcal{V}(w) \end{array}$$

Here a bijective map

$$\iota : (P^w \cap L') \backslash L' / L'_K \rightarrow (P^w \cap P') \backslash L'U' / L'_K U' \quad (2.4)$$

is induced by the inclusion $L' \hookrightarrow L'U'$ and the second vertical arrow in the diagram is the inverse of ι . The bijectivity of ι is deduced from the following general lemma.

Lemma 2.4. *Let $L_1 \ltimes U_1$ be a semidirect product group of two groups L_1 and U_1 . Let $L_2, L_3 \subset L_1$ and $U_2 \subset U_1$ be subgroups and assume that L_2 normalizes U_2 so that $L_2 \ltimes U_2$ is a subgroup of $L_1 \ltimes U_1$. Then the natural inclusion map induces the following bijections:*

$$L_2 \backslash L_1 / L_3 \xrightarrow{\sim} L_2 \backslash (L_1 U_1) / (L_3 U_1) \xrightarrow{\sim} (L_2 U_2) \backslash (L_1 U_1) / (L_3 U_1).$$

Proof. It is easy to see that the both maps are well-defined and surjective. So it is enough to see that the composite map is injective.

Suppose $l, l' \in L_1$ satisfy $L_2 U_2 l L_3 U_1 = L_2 U_2 l' L_3 U_1$. This means that there exist $l_2 \in L_2, l_3 \in L_3, u_2 \in U_2, u_1 \in U_1$ such that $l' = (l_2 u_2) l (l_3 u_1)$, or equivalently $l' = (l_2 l l_3) (((l l_3)^{-1} u_2 (l l_3)) u_1) \in L_1 \ltimes U_1$. By the uniqueness of the semidirect product decomposition, we have $l' = l_2 l l_3$ and hence $l' \in L_2 l L_3$. \square

To see that the map ι in (2.4) is bijective, we use Lemma 2.4 in the setting where $L_1 = L', U_1 = U', L_2 = P^w \cap L', U_2 = P^w \cap U', L_3 = L'_K$. Note that

$$P^w \cap P' = (P^w \cap L') \ltimes (P^w \cap U')$$

holds (see [Car85, Theorem 2.8.7 and Proposition 2.8.9]). \square

Let us summarize the above situation into a diagram:

$$\begin{array}{ccccccc} P w a Q & \in & P \backslash P w P' / Q & \xrightarrow{\sim} & (P^w \cap P') \backslash P' / Q & \ni & (P^w \cap P') a Q \\ \downarrow & & \Psi_w \downarrow & & \downarrow \text{projection} & & \downarrow \\ (P^w \cap L') b L'_K & \in & (P^w \cap L') \backslash L' / L'_K & \xrightarrow{\sim} & (P^w \cap P') \backslash P' / Q U' & \ni & (P^w \cap P') b Q U' \end{array}$$

where $a = bu \in L' U'$ is the Levi decomposition. Let us take representatives of $\mathcal{V}(w) = (P^w \cap L') \backslash L' / L'_K$ from L' and identify them with $\mathcal{V}(w)$ in the following. We would like to analyze the fiber $\Psi_w^{-1}(v) = P \backslash P w v U' Q / Q$ for $v \in \mathcal{V}(w)$.

2.3. Parameterization of orbits on the double flag variety. Now we get a rough parameterization of orbits, first by the Bruhat decomposition $P \backslash G / P' \simeq {}^J W^{J'}$, then next by the KGB decomposition for the smaller symmetric space L' / L'_K .

The following lemma describes the fiber $\Psi_w^{-1}(v) = P \backslash P w v U' Q / Q$ for $v \in \mathcal{V}(w)$.

Lemma 2.5. *Let $w \in {}^J W^{J'}$ and $v \in \mathcal{V}(w)$, which are identified with their representatives.*

- (1) $\Psi_w^{-1}(v) = P \backslash P w v U' Q / Q \simeq (P^{wv} \cap P') \backslash (P^{wv} \cap P') U' Q / Q$.
- (2) *We can define the following surjective map:*

$$\begin{array}{ccc} (P^{wv} \cap U') \backslash U' / U'_K & \longrightarrow & (P^{wv} \cap P') \backslash (P^{wv} \cap P') U' Q / Q \\ (P^{wv} \cap U') u U'_K & \longmapsto & (P^{wv} \cap P') u Q. \end{array}$$

(3) The above surjection factors through to a bijection

$$\left((P^{wv} \cap U') \backslash U' / U'_K \right) / (P^{wv} \cap L'_K) \xrightarrow{\sim} (P^{wv} \cap P') \backslash (P^{wv} \cap P') U' Q / Q.$$

Notice that $P^{wv} \cap L'_K$ normalizes $P^{wv} \cap U'$ and U'_K , hence the conjugation action of $P^{wv} \cap L'_K$ on U' induces an action on $(P^{wv} \cap U') \backslash U' / U'_K$. The corresponding quotient space is the one we considered above.

Proof. (1) This follows from Lemma 2.2.

(2) Since $P^{wv} \cap U' \subset P^{wv} \cap P'$ and $U'_K \subset Q$, our map is just a projection.

(3) We use the following general lemma, in which the notations are independent of the rest of the arguments.

Lemma 2.6. *Let $L_1 \ltimes U_1$ be a semidirect product group of two groups L_1 and U_1 . Let $L_2, L_3 \subset L_1$ and $U_2, U_3 \subset U_1$ be subgroups and assume that L_i normalizes U_i for $i = 2, 3$ so that $L_2 \ltimes U_2$ and $L_3 \ltimes U_3$ are subgroups of $L_1 \ltimes U_1$.*

(1) *The conjugation action of the group $L_2 \cap L_3$ on U_1 by $u \mapsto l u l^{-1}$ induces a well-defined action of $L_2 \cap L_3$ on $U_2 \backslash U_1 / U_3$.*

(2) *The natural map $U_2 \backslash U_1 / U_3 \rightarrow (L_2 U_2) \backslash (L_2 U_1 L_3) / (L_3 U_3)$, $U_2 u U_3 \mapsto (L_2 U_2) u (L_3 U_3)$ induces a bijective map*

$$\varphi : (U_2 \backslash U_1 / U_3) / (L_2 \cap L_3) \xrightarrow{\sim} (L_2 U_2) \backslash (L_2 U_1 L_3) / (L_3 U_3).$$

Proof. The claim (1) is obvious.

Let us prove (2). The surjectivity of φ is clear since we can always take a representative of the right-hand side in U_1 . We give a proof of injectivity. For $u, u' \in U_1$, let us assume that $(L_2 U_2) u (L_3 U_3) = (L_2 U_2) u' (L_3 U_3)$. Then there exist $l_2 u_2 \in L_2 U_2$ and $l_3 u_3 \in L_3 U_3$ such that $u' = (l_2 u_2) u (l_3 u_3)$. We rewrite it as

$$u' = (l_2 l_3) (l_3^{-1} u_2 u l_3) u_3 \in L_1 U_1.$$

By the uniqueness of the semidirect product, $l_2 l_3 = e$ and $u' = (l_3^{-1} u_2 u l_3) u_3$. Therefore, we have $l_2 = l_3^{-1} \in L_2 \cap L_3$ and

$$u' = (l_2 u_2 l_2^{-1}) (l_2 u l_2^{-1}) u_3 \in U_2 (l_2 u l_2^{-1}) U_3.$$

This shows $u' \in (L_2 \cap L_3) \cdot (U_2 u U_3)$, where \cdot denotes the conjugation action. \square

To prove Lemma 2.5 (3), we apply Lemma 2.6 (2) in the setting where

$$L_1 = L', \quad L_2 = P^{wv} \cap L', \quad L_3 = L'_K, \quad U_1 = U', \quad U_2 = P^{wv} \cap U', \quad U_3 = U'_K.$$

We need to use again $P^{wv} \cap P' = (P^{wv} \cap L') \ltimes (P^{wv} \cap U')$ ([Car85, Theorem 2.8.7 and Proposition 2.8.9]), and $Q = L'_K U'_K$. \square

Lemma 2.5 with two reductions (§§ 2.1 and 2.2) gives us a parameterization of K -orbits on the double flag variety $\mathfrak{X}_P \times \mathfrak{Z}_Q$.

Theorem 2.7. *Let $P = P_J$ and $P' = P_{J'}$ be standard parabolic subgroups of G and assume that P' is θ -stable with the standard (θ -stable) Levi decomposition $P' = L'U'$. Define $Q := P' \cap K$, which is a parabolic subgroup of K , and put $L'_K := L' \cap K$, $U'_K := U' \cap K$.*

(1) *The K -orbits on the double flag variety $\mathfrak{X}_P \times \mathfrak{Z}_Q = G/P \times K/Q$ are parameterized as follows:*

$$K \backslash (\mathfrak{X}_P \times \mathfrak{Z}_Q) \simeq \coprod_{w \in {}^J W^{J'}} \coprod_{v \in \mathcal{V}(w)} \left((P^{wv} \cap U') \backslash U' / U'_K \right) / P^{wv} \cap L'_K,$$

Here we write ${}^J W^{J'} := W_J \backslash W / W_{J'}$, $\mathcal{V}(w) := (P^w \cap L') \backslash L' / L'_K$ and identify them with their representatives.

(2) *The double flag variety $\mathfrak{X}_P \times \mathfrak{Z}_Q$ is of finite type if and only if for any $w \in {}^J W^{J'}$ and $v \in \mathcal{V}(w)$, the conjugation action of $P^{wv} \cap L'_K$ on the double coset space $(P^{wv} \cap U') \backslash U' / U'_K$ has only finitely many orbits.*

Proof. The claim (1) was already proved. Since ${}^J W^{J'}$ is a finite set and $\mathcal{V}(w)$ is also finite for any $w \in {}^J W^{J'}$, the claim (2) follows. \square

Corollary 2.8. *The double flag variety $\mathfrak{X}_P \times \mathfrak{Z}_Q$ is of finite type if and only if for any $g \in P W L'$, the conjugation action of $P^g \cap L'_K$ on $(P^g \cap U') \backslash U' / U'_K$ has only finitely many orbits.*

Proof. This is obvious from Theorem 2.7 once one knows

$$\bigcup_{w \in {}^J W^{J'}} \bigcup_{v \in \mathcal{V}(w)} P w v L'_K = \bigcup_{w \in {}^J W^{J'}} P w L' = P W L'.$$

\square

3. LINEARIZATION OF UNIPOTENT DOUBLE COSET SPACES

In this section, we study the unipotent double coset spaces appearing in Theorem 2.7. We will prove that the double coset space can be reduced to the quotient space of a linear space by a linear action under certain assumptions.

Let U be a unipotent group on which the torus $T^1 = \mathbb{C}^\times$ acts by group automorphisms. Let us denote by $\rho : T^1 \rightarrow \text{Aut } U$ the given action. Then T^1 acts on the Lie algebra $\mathfrak{u} = \text{Lie } U$ by differential, which we also denote by the same letter ρ .

We assume the following throughout this section:

Assumption 3.1. *The weights of T^1 on \mathfrak{u} are all positive.*

Lemma 3.2. *Let $\mathfrak{u} = \mathfrak{u}_1 \oplus \cdots \oplus \mathfrak{u}_n$ be a decomposition of \mathfrak{u} as a T^1 -module. Then the map*

$$\begin{aligned} \varphi : \mathfrak{u}_1 \oplus \cdots \oplus \mathfrak{u}_n &\xrightarrow{\sim} U \\ (X_1, \dots, X_n) &\longmapsto \exp X_1 \cdots \exp X_n \end{aligned}$$

is bijective.

Proof. By the standard arguments of Lie theory, we can take neighborhoods (with respect to the classical topology) $0 \in \mathfrak{w}_i \subset \mathfrak{u}_i$ for $i = 1, \dots, n$ and $e \in W \subset U$ such that the restriction of φ to $\mathfrak{w}_1 \times \dots \times \mathfrak{w}_n$ gives a homeomorphism

$$\begin{aligned} \mathfrak{w}_1 \times \dots \times \mathfrak{w}_n &\xrightarrow{\approx} W \subset U \\ (X_1, \dots, X_n) &\longmapsto \exp X_1 \cdots \exp X_n. \end{aligned}$$

Let us prove that φ is surjective. Take $u \in U$. Then there exists $t \in T^1$ such that $\rho(t)u \in W$. Indeed, since U is unipotent, the exponential map $\exp : \mathfrak{u} \rightarrow U$ is a T^1 -equivariant isomorphism. So we can find $Z \in \mathfrak{u}$ such that $u = \exp Z$. Since the weights of T^1 are all positive, there exists $t \in T^1$ such that $\rho(t)Z$ is small enough to be contained in $\log W$, where \log is the inverse map of \exp . Thus we get $\rho(t)u = \rho(t) \exp Z = \exp(\rho(t)Z) \in W$. Now we can choose $(Z_1, \dots, Z_n) \in \mathfrak{w}_1 \times \dots \times \mathfrak{w}_n$ such that $\varphi(Z_1, \dots, Z_n) = \rho(t)u$. From this, we conclude that $\varphi(\rho(t^{-1})Z_1, \dots, \rho(t^{-1})Z_n) = u$.

The injectivity can be proved in a similar way. \square

Let U and T^1 be as above. Let $U_1, U_2 \subset U$ be subgroups of U which are stable under the action of T^1 . Take a decomposition $\mathfrak{u} = (\mathfrak{u}_1 + \mathfrak{u}_2) \oplus \mathfrak{V}$ as a T^1 -module. We further assume that \mathfrak{V} is stable under the adjoint action of $U_1 \cap U_2$.

Proposition 3.3. *Under the above notations and assumptions, the map*

$$\Phi : \mathfrak{V}/(U_1 \cap U_2) \rightarrow U_1 \backslash U / U_2$$

given by $(U_1 \cap U_2) \cdot Z \mapsto U_1(\exp Z)U_2$ is bijective. Here $U_1 \cap U_2$ acts on \mathfrak{V} by the adjoint action.

Proof. We take a T^1 -stable complementary subspace \mathfrak{w}_i ($i = 1, 2$) of $\mathfrak{u}_1 \cap \mathfrak{u}_2$ in \mathfrak{u}_i so that $\mathfrak{u}_i = \mathfrak{w}_i \oplus (\mathfrak{u}_1 \cap \mathfrak{u}_2)$ is a decomposition of a T^1 -module. Then we have a decomposition of \mathfrak{u} as

$$\mathfrak{u} = \mathfrak{w}_1 \oplus \mathfrak{V} \oplus (\mathfrak{u}_1 \cap \mathfrak{u}_2) \oplus \mathfrak{w}_2.$$

By applying Lemma 3.2 to this decomposition, we see that every element $u \in U$ is uniquely written as $u = u_1(\exp Z)u_3u_2$ where $u_1 \in \exp \mathfrak{w}_1$, $Z \in \mathfrak{V}$, $u_3 \in U_1 \cap U_2$, and $u_2 \in \exp \mathfrak{w}_2$. Therefore, the map Φ is surjective. To prove the injectivity, suppose that $\exp Z_1 = u_1(\exp Z_2)u_2$ for $Z_1, Z_2 \in \mathfrak{V}$ and $u_i \in U_i$ ($i = 1, 2$). By Lemma 3.2 again, we have $u_1 = u''_1 u'_1$ where $u''_1 \in \exp(\mathfrak{w}_1)$ and $u'_1 \in \exp(\mathfrak{u}_1 \cap \mathfrak{u}_2) = U_1 \cap U_2$. Also we can write $u_2 = u'_2 u''_2$ where $u'_2 \in U_1 \cap U_2$ and $u''_2 \in \exp(\mathfrak{w}_2)$. Then we can compute as

$$\begin{aligned} u_1(\exp Z_2)u_2 &= u''_1 u'_1(\exp Z_2)u'_2 u''_2 \\ &= u''_1(\exp \text{Ad}(u'_1)Z_2)(u'_1 u'_2)u''_2 \in \exp \mathfrak{w}_1 \exp \mathfrak{V} \exp(\mathfrak{u}_1 \cap \mathfrak{u}_2) \exp \mathfrak{w}_2. \end{aligned}$$

Since the decomposition is unique, we get $Z_1 = \text{Ad}(u'_1)Z_2$, which shows Z_1 and Z_2 are in the same $\text{Ad}(U_1 \cap U_2)$ -orbit. \square

In general, we cannot apply Proposition 3.3 directly to Theorem 2.7 (1) because our assumptions do not hold. However, there are some special cases where we can apply it, which we will discuss in the following sections.

4. SPHERICAL ACTION OF A SYMMETRIC SUBGROUP ON A PARTIAL FLAG VARIETY

Under the setting of § 2, we now assume that $P' = B$ so that $Q = B \cap K =: B_K$ is a Borel subgroup of K and the double flag variety is $\mathfrak{X}_P \times \mathcal{Z}_{B_K} = G/P \times K/B_K$.

We take a θ -stable maximal torus T as in § 1, and denote by $B = TU_B$ a Levi decomposition of B (U_B denotes the unipotent radical of B). In our former notation,

$$\begin{aligned} P' &= B = TU_B = L'U', \\ Q &= B_K = T_K U_{B_K} = L'_K U'_K \quad (T_K = T \cap K, U_{B_K} = U_B \cap K), \\ {}^J W^{J'} &= {}^J W^\emptyset =: {}^J W \ni w, \\ P^w \cap L' &= P^w \cap T = T \quad (\text{for any } w \in {}^J W), \\ \mathcal{V}(w) &= (P^w \cap L') \backslash L' / L'_K = T \backslash T / T_K = \{e\}, \\ P^{wv} \cap L'_K &= T_K, \quad P^w \cap U' = P^w \cap U_B. \end{aligned}$$

Then Theorem 2.7 (1) in this case can be written as follows.

Theorem 4.1. *Let $P = P_J$ be a standard parabolic subgroup of G and B_K a Borel subgroup of K . Then the K -orbits on the double flag variety are parameterized as follows:*

$$K \backslash (\mathfrak{X}_P \times \mathcal{Z}_{B_K}) \simeq \coprod_{w \in {}^J W} \left((w^{-1} P w \cap U_B) \backslash U_B / U_{B_K} \right) / T_K. \quad (4.1)$$

Remark 4.2. If $\text{rank } G = \text{rank } K$, then we have $T = T_K$.

Since B_K is a Borel subgroup of K , the flag variety G/P has only finitely many B_K -orbits if and only if it has an open dense B_K -orbit, i.e.,

$$\begin{aligned} \mathfrak{X}_P \times \mathcal{Z}_{B_K} &\text{ is of finite type} \\ \iff \mathfrak{X}_P = G/P &\text{ is } K\text{-spherical} \\ \iff &\text{ there exists an open dense } B_K\text{-orbit on } \mathfrak{X}_P \\ \iff &\text{ there exists an open dense } K\text{-orbit on } \mathfrak{X}_P \times \mathcal{Z}_{B_K}. \end{aligned}$$

Thus to see whether the double flag variety is of finite type, we can concentrate on the fiber of the longest element of ${}^J W := W_J \backslash W$, which corresponds to the open stratum of the Bruhat decomposition $P \backslash G / B$. Denote by $w_0 \in W$ the longest element in the Weyl group and put

$$\begin{aligned} J^\star &:= -w_0(J) \subset \Pi, \\ P_{J^\star} &= L_{J^\star} U_{J^\star} : \text{Levi decomposition,} \\ w_0^{-1} P w_0 \cap U_B &= L_{J^\star} \cap U_B = U_{L_{J^\star}} : \text{a maximal unipotent subgroup of } L_{J^\star}. \end{aligned} \quad (4.2)$$

With this notation, we can state an immediate consequence of Theorem 4.1.

Corollary 4.3. *$\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ is of finite type if and only if the conjugation action of T_K on $U_{L_{J^*}} \backslash U_B / U_{B_K}$ has only finitely many orbits.*

The condition of the corollary above reduces to a linear action by using Proposition 3.3. Now let us prove the following key lemma.

Lemma 4.4. (1) *The group $L_{J^*} \cap K = L_{J^* \cap \theta(J^*)} \cap K$ is a connected reductive group, and it acts on $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$ by the conjugation action.*

(2) *The exponential map induces an $(L_{J^*} \cap K)$ -equivariant bijective map*

$$(\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}) / (B_K \cap L_{J^*}) \xrightarrow{\sim} \left((U_B \cap L_{J^*}) \backslash U_B / U_{B_K} \right) / T_K,$$

where $B_K \cap L_{J^*}$ acts on $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$ by the adjoint action.

(3) *$B_K \cap L_{J^*}$ is a Borel subgroup of $L_{J^*} \cap K$.*

Proof. (1) It follows that $L_{J^*} \cap K = L_{J^*} \cap L_{\theta(J^*)} \cap K = L_{J^* \cap \theta(J^*)} \cap K$. Then $L_{J^* \cap \theta(J^*)} \cap K$ is a Levi component of the parabolic subgroup $P_{J^* \cap \theta(J^*)} \cap K$ of K . Hence it is connected and reductive. Since L_{J^*} normalizes $\mathfrak{u}_{P_{J^*}}$ and K normalizes $\mathfrak{g}^{-\theta}$, the intersection $L_{J^*} \cap K$ acts on $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$.

(2) We use Proposition 3.3 by taking $U = U_B$, $U_1 = U_B \cap L_{J^*}$ and $U_2 = U_{B_K}$. Since T_K contains a regular element of G , we can take a subtorus $T^1 = \mathbb{C}^\times$ in T_K so that the weights in $\mathfrak{u} = \mathfrak{u}_B$ are all positive. Then $\mathfrak{u}_B \cap \mathfrak{l}_{J^*}$ and $\mathfrak{u}_{B_K} = \mathfrak{u}_B^\theta$ are stable under T^1 . We have

$$\begin{aligned} \mathfrak{u}_1 &= \mathfrak{u}_B \cap \mathfrak{l}_{J^*}, \\ \mathfrak{u}_2 &= \mathfrak{u}_{B_K} = \mathfrak{u}_B^\theta, \\ \mathfrak{u}_1 \cap \mathfrak{u}_2 &= (\mathfrak{u}_B \cap \mathfrak{l}_{J^*}) \cap \mathfrak{u}_{B_K} = \mathfrak{u}_B^\theta \cap \mathfrak{l}_{J^*} = \mathfrak{u}_B^\theta \cap (\mathfrak{l}_{J^*} \cap \mathfrak{l}_{\theta(J^*)}), \\ \mathfrak{u}_1 + \mathfrak{u}_2 &= (\mathfrak{u}_B \cap \mathfrak{l}_{J^*}) + \mathfrak{u}_B^\theta = (\mathfrak{u}_B \cap (\mathfrak{l}_{J^*} + \mathfrak{l}_{\theta(J^*)})) + \mathfrak{u}_B^\theta, \\ \mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta} &= \mathfrak{u}_{P_{J^*}} \cap \mathfrak{u}_{P_{\theta(J^*)}} \cap \mathfrak{g}^{-\theta}, \\ \mathfrak{u} &= \mathfrak{u}_B = (\mathfrak{u}_B \cap (\mathfrak{l}_{J^*} + \mathfrak{l}_{\theta(J^*)})) \oplus (\mathfrak{u}_{P_{J^*}} \cap \mathfrak{u}_{P_{\theta(J^*)}}) = (\mathfrak{u}_1 + \mathfrak{u}_2) \oplus (\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}). \end{aligned}$$

Here $\mathfrak{u}_{P_{J^*}}$ is the nilradical of $\mathfrak{p}_{J^*} = \text{Lie } P_{J^*}$. Since $U_1 \cap U_2 = U_{B_K} \cap L_{J^*}$ is contained in $K \cap L_{J^*}$, it acts on $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$ via the adjoint action by (1). Now we take $\mathfrak{V} = \mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$ in Proposition 3.3 and conclude that the exponential map induces a bijective map

$$(\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}) / (U_{B_K} \cap L_{J^*}) \xrightarrow{\sim} (U_B \cap L_{J^*}) \backslash U_B / U_{B_K}.$$

The torus T_K acts on the both hand sides by the adjoint (or conjugation) action and we see that

$$(\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}) / (B_K \cap L_{J^*}) \xrightarrow{\sim} \left((U_B \cap L_{J^*}) \backslash U_B / U_{B_K} \right) / T_K,$$

which is the claim in (2).

(3) We know $B_K \cap L_{J^*} = (B \cap L_{J^* \cap \theta(J^*)}) \cap K$ by (1). It is clear that $B \cap L_{J^* \cap \theta(J^*)}$ is a θ -stable Borel subgroup of $L_{J^* \cap \theta(J^*)}$. So it cuts out a Borel subgroup of $(L_{J^* \cap \theta(J^*)})^\theta = L_{J^* \cap \theta(J^*)} \cap K = L_{J^*} \cap K$. \square

By Corollary 4.3, the double flag variety $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ is of finite type if and only if

$$\# \left((U_B \cap L_{J^*}) \backslash U_B / U_{B_K} \right) / T_K < \infty \quad (4.3)$$

holds. On the other hand, by Lemma 4.4, there is a bijection between the T_K -orbit space of the double cosets $(U_B \cap L_{J^*}) \backslash U_B / U_{B_K}$ and the quotient space $(\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}) / (B_K \cap L_{J^*})$. Note that the reductive group $(L_{J^*})^\theta = L_{J^*} \cap K$ acts on $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$ by the adjoint action. Since $B_K \cap L_{J^*}$ is a Borel subgroup of $L_{J^*} \cap K$, the finiteness of orbits in (4.3) is equivalent to that $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$ is $(L_{J^*} \cap K)$ -spherical.

Note that $P_{J^*} \cap \theta(P_{J^*}) = P_{J^* \cap \theta(J^*)}$ is a θ -stable parabolic subgroup. Since $P_{J^*} \cap K = P_{J^* \cap \theta(J^*)} \cap K$ holds, P_{J^*} cuts a parabolic subgroup from K . Thus the K -translates of P_{J^*} in $\mathfrak{X}_{P_{J^*}}$ form a closed K -orbit $\mathcal{O} \simeq K / (P_{J^*} \cap K)$. Let us consider the conormal bundle over \mathcal{O} in the cotangent bundle $T^*\mathfrak{X}_{P_{J^*}}$, which is denoted by $T_{\mathcal{O}}^*\mathfrak{X}_{P_{J^*}}$. Then the fiber of this conormal bundle at the base point $P_{J^*} \in \mathfrak{X}_{P_{J^*}}$ is $(\mathfrak{g} / (\mathfrak{p}_{J^*} + \mathfrak{k}))^*$, which is identified with $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta} = (\mathfrak{u}_{P_{J^*}} \cap \mathfrak{u}_{P_{\theta(J^*)}})^{-\theta}$ via a non-degenerate invariant bilinear form on \mathfrak{g} . With this notation we see

$$T_{\mathcal{O}}^*\mathfrak{X}_{P_{J^*}} \simeq K \times_{P_{J^*} \cap K} (\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}).$$

If $k \in K$ is a representative of the longest element of the Weyl group for K , then $(B_K)^k(P_{J^*} \cap K)$ is open in K and $(B_K)^k \cap P_{J^*}$ is a Borel subgroup of $L_{J^*} \cap K$. Here $(B_K)^k := k^{-1}B_Kk$. Therefore, a Borel subgroup of $L_{J^*} \cap K$ has an open orbit on $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$ if and only if the conormal bundle $T_{\mathcal{O}}^*\mathfrak{X}_{P_{J^*}}$ is K -spherical.

We summarize the results into the following theorem.

Theorem 4.5. *Let B_K be a Borel subgroup of K and $P = P_J$ a parabolic subgroup of G . Put $J^* = -w_0(J)$, where w_0 is the longest element in the Weyl group W . Let $\mathcal{O} \subset \mathfrak{X}_{P_{J^*}}$ be the closed K -orbit through the base point $P_{J^*} \in \mathfrak{X}_{P_{J^*}}$. Then the following five conditions are equivalent.*

- (1) *The double flag variety $\mathfrak{X}_P \times \mathcal{Z}_{B_K} = G/P \times K/B_K$ is of finite type.*
- (2) *The adjoint action of $L_{J^*} \cap K$ on $\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta}$ is spherical.*
- (3) *The coadjoint action of $L_{J^*} \cap K$ on $(\mathfrak{u}_{P_{J^*}} \cap \mathfrak{g}^{-\theta})^*$ is spherical.*
- (4) *The conormal bundle $T_{\mathcal{O}}^*\mathfrak{X}_{P_{J^*}}$ is K -spherical.*
- (5) *The normal bundle $T_{\mathcal{O}}\mathfrak{X}_{P_{J^*}}$ is K -spherical.*

Proof. We have already explained the equivalence of (1), (2), and (4). The equivalence of (3) and (5) can be proved in a similar way. The equivalence of (2) and (3) follows from Lemma 1.1. \square

We can employ a similar argument by using the opposite Borel subgroup instead of B . Let B^- be the Borel subgroup corresponding to the negative roots $-\Delta^+$ and U_B^- its unipotent radical. Then it turns out that G/P is K -spherical if and only if $((P \cap U_B^-) \backslash U_B^- / (U_B^- \cap K)) / T_K$ is finite. Define \mathfrak{u}_P^- as the nilradical of the opposite parabolic subalgebra of \mathfrak{p} so that $\mathfrak{g} = \mathfrak{u}_P^- \oplus \mathfrak{p}$. Then we can conclude that $((P \cap U_B^-) \backslash U_B^- / (U_B^- \cap K)) / T_K$ is finite if and only if $\mathfrak{u}_P^- \cap \mathfrak{g}^{-\theta}$ is $(L \cap K)$ -spherical. Let \mathcal{O} be the closed K -orbit in \mathfrak{X}_P through P . Then the fiber of normal bundle $T_{\mathcal{O}}\mathfrak{X}_P$ at $P \in \mathfrak{X}_P$ is isomorphic to $\mathfrak{g}/(\mathfrak{p} + \mathfrak{k})$ and it is identified with $\mathfrak{u}_P^- \cap \mathfrak{g}^{-\theta}$ via an invariant form on \mathfrak{g} . As a consequence, \mathfrak{X}_P is K -spherical if and only if $T_{\mathcal{O}}\mathfrak{X}_P$ is K -spherical.

Now we can conclude:

Theorem 4.6. *Let us use the same notation as in Theorem 4.5. Then the following are all equivalent.*

- (1) *The double flag variety $\mathfrak{X}_P \times \mathcal{Z}_{B_K} = G/P \times K/B_K$ is of finite type.*
- (2) *$\mathfrak{X}_P = G/P$ is K -spherical.*
- (3) *$\mathfrak{X}_{P_{J^*}} = G/P_{J^*}$ is K -spherical.*
- (4) *The adjoint action of $L \cap K$ on $\mathfrak{u}_P^- \cap \mathfrak{g}^{-\theta}$ is spherical.*
- (5) *For the closed K -orbit $\mathcal{O} \subset \mathfrak{X}_P$ through the base point $P \in \mathfrak{X}_P$, the normal bundle $T_{\mathcal{O}}\mathfrak{X}_P$ is K -spherical.*

Proof. We saw above the equivalence of (1), (2), (4), and (5). The equivalence of (3) and (5) follows from Theorem 4.5. \square

Remark 4.7. The equivalence of (1) and (5) can be also deduced from [Pan99, Theorem 2.1].

We note that actions on the fiber of a conormal bundle has much to do with Springer fibers. See [BZ08] and [SY05].

5. SPHERICAL FIBER BUNDLE OVER AFFINE SYMMETRIC SPACE

In this section, we consider the case where $P = B$ is a Borel subgroup so $J = \emptyset$ in our previous notation and the double flag variety is $\mathfrak{X}_B \times \mathcal{Z}_Q = G/B \times K/Q$. Recall that we have assumed that B is θ -stable.

We summarize the notation here, which is adapted to the present situation.

$${}^J W^{J'} = W^{J'} \ni w,$$

$$P^w \cap L' = w^{-1} B w \cap L' : \text{a Borel subgroup of } L',$$

$$\mathcal{V}(w) = (P^w \cap L') \backslash L' / L'_K = (w^{-1} B w \cap L') \backslash L' / L'_K \ni v : \text{a representative in } L',$$

$$P^{wv} \cap L'_K = (L' \cap (wv)^{-1} B wv) \cap K, \quad \text{where } L' \cap (wv)^{-1} B wv \text{ runs over all the Borel subgroups in } L' \text{ up to } L'_K\text{-conjugacy,}$$

$$P^{wv} \cap U' = (wv)^{-1} B wv \cap U' = v^{-1} (w^{-1} B w \cap U') v = v^{-1} (U' \cap w^{-1} U_B w) v.$$

Thus we have

$$\begin{aligned} & \left((P^{wv} \cap U') \backslash U' / U'_K \right) / (P^{wv} \cap L'_K) \\ & \simeq \left((U' \cap (wv)^{-1} U_B wv) \backslash U' / U'_K \right) / (L' \cap (wv)^{-1} B wv \cap K), \end{aligned}$$

where $U'_K := U' \cap K$. So Theorem 2.7 (1) becomes

$$K \backslash (\mathfrak{X}_B \times \mathcal{Z}_Q) \simeq \coprod_{w \in W^{J'}} \coprod_{v \in \mathcal{V}(w)} \left((U' \cap (wv)^{-1} U_B wv) \backslash U' / U'_K \right) / (L' \cap (wv)^{-1} B wv \cap K).$$

Since $\mathfrak{X}_B \times \mathcal{Z}_Q$ is of finite type if and only if G/Q is G -spherical, we concentrate on the existence of an open B -orbit in G/Q . Take the longest element $w_0 \in W$ so that

$$U' \cap w_0^{-1} U_B w_0 = \{e\} \quad \text{and} \quad L' \cap w_0^{-1} B w_0 = L' \cap B^-,$$

where B^- denotes the opposite Borel subgroup corresponding to $\Delta^- = -\Delta^+$. Therefore, we get

$$(P^{w_0 v} \cap U') \backslash U' / U'_K = U' / U'_K \quad \text{and} \quad P^{w_0 v} \cap L'_K = v^{-1} (L' \cap B^-) v \cap K.$$

Since $\mathcal{V}(w_0) = (L' \cap B^-) \backslash L' / L'_K$ and $(L' \cap B^-) \backslash L'$ is the full flag variety of L' , the subgroup $v^{-1} (L' \cap B^-) v$ ($v \in \mathcal{V}(w)$) runs over all the Borel subgroups of L' up to L'_K -conjugacy.

Let us recall some general facts about the double coset space $B \backslash G / K$. Afterward, we will apply it to the double coset space $(L' \cap B^-) \backslash L' / L'_K$.

A parabolic subgroup P of G is called θ -split if $P \cap \theta(P)$ is a Levi component of P . It is known that there exists uniquely a minimal θ -split parabolic subgroup of G up to K -conjugacy. Let P_{\min} be a minimal θ -split parabolic subgroup of G and we take a θ -stable maximal torus T in P_{\min} . Put $A := \exp \mathfrak{t}^{-\theta}$ and $M := Z_K(A)$, the centralizer of A in K . Then it follows that $P_{\min} \cap K = M$ and $P_{\min} = MAN$, where N is the unipotent radical of P_{\min} . The following lemma is well-known (see [HW93, Proposition 9.2] for general case).

Lemma 5.1. *For $v \in G$, the set BvK is open in G if and only if there exists a minimal θ -split parabolic subgroup P_{\min} that contains $v^{-1}Bv$.*

If $v^{-1}Bv \subset P_{\min}$, then $v^{-1}Bv \cap K = (v^{-1}Bv \cap M)AN$ and the identity component of $v^{-1}Bv \cap M$ is a Borel subgroup of the identity component of M .

Now we specialize Theorem 2.7 to the present case. Since \mathfrak{u}' is θ -stable, we have a decomposition $\mathfrak{u}' = \mathfrak{u}'_K \oplus (\mathfrak{u}')^{-\theta}$. Therefore, we get an L'_K -equivariant isomorphism $(\mathfrak{u}')^{-\theta} \simeq U' / U'_K$ via the exponential map.

Theorem 5.2. *Let P' be a θ -stable parabolic subgroup of G such that $P' \cap K = Q$. We denote by $P' = L'U'$ the standard Levi decomposition. Let P'_{\min} be a minimal θ -split parabolic subgroup of L' and put $M' := P'_{\min} \cap K$. Then the following four conditions are all equivalent.*

- (1) *The double flag variety $\mathfrak{X}_B \times \mathcal{Z}_Q = G/B \times K/Q$ is of finite type.*

- (2) *The homogeneous space G/Q is G -spherical.*
- (3) *$U'/(U' \cap K)$ has only finitely many $(B_{L'} \cap K)$ -orbits for any Borel subgroup $B_{L'}$ of L' .*
- (4) *The adjoint action of the identity component M'_0 of M' on $(\mathfrak{u}')^{-\theta}$ is spherical.*

Proof. The equivalence of (1), (2) and (3) has already been proved.

Let us take a Borel subgroup $B_{L'} \subset L'$ which is contained in P'_{\min} . Then the identity component of $B_{L'} \cap K$ is a Borel subgroup of M'_0 . Since $(\mathfrak{u}')^{-\theta}$ is L'_K -equivariantly isomorphic to $U'/(U' \cap K)$, the condition (3) implies (4).

Conversely, the condition (4) implies (3) for the case where $B_{L'}$ corresponds to the open L'_K -orbit of $L'/(L' \cap B^-)$. Hence it shows the existence of an open B -orbit in G/Q . \square

Corollary 5.3. *If the double flag variety $\mathfrak{X}_B \times \mathcal{Z}_Q$ is of finite type, then the double flag variety $\mathfrak{X}_{P'} \times \mathcal{Z}_{B_K}$ is also of finite type.*

Proof. By Theorem 5.2, the adjoint action of M'_0 on $(\mathfrak{u}')^{-\theta}$ is spherical. Since $M' \subset L'_K$, we conclude that $\mathfrak{X}_{P'} \times \mathcal{Z}_{B_K}$ is of finite type by Theorem 4.6. \square

6. TRIPLE FLAG VARIETIES

Let us take three parabolic subgroups P_1, P_2 and P_3 of G . If one considers $\mathbb{G} = G \times G$ and an involution $\theta(g_1, g_2) = (g_2, g_1)$ of \mathbb{G} , the symmetric subgroup $\mathbb{K} = \mathbb{G}^\theta$ is the diagonal subgroup $\text{diag}(G) \subset \mathbb{G}$. Thus $(G \times G, \text{diag}(G))$ is a symmetric pair. Then $\mathbb{P} = P_1 \times P_2$ is a parabolic subgroup of \mathbb{G} and $\mathbb{Q} = \text{diag}(P_3)$ is a parabolic subgroup of \mathbb{K} . Therefore our double flag variety becomes

$$\mathbb{G}/\mathbb{P} \times \mathbb{K}/\mathbb{Q} = (G \times G)/(P_1 \times P_2) \times (\text{diag}(G)/\text{diag}(P_3)) \simeq \mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3},$$

which is a triple flag variety for G . So the double flag variety for symmetric pair is a generalization of the triple flag variety. We can take a parabolic subgroup $\mathbb{P}' = P_3 \times P_3$ of \mathbb{G} so that $\mathbb{P}' \cap \mathbb{K} = \text{diag}(P_3) = \mathbb{Q}$ holds.

A triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$ is said to be of finite type if there are only finitely many G -orbits in it. Note that this terminology agrees with that of the double flag variety for symmetric pair.

If T is a maximal torus of G , then $\mathbb{T} = T \times T$ is a maximal torus of \mathbb{G} . The root system of \mathbb{G} with respect to this \mathbb{T} is decomposed as $\Delta_{\mathbb{G}} = \Delta^{(1)} \sqcup \Delta^{(2)}$, where $\Delta^{(1)}$ denotes the roots of the first factor and $\Delta^{(2)}$ denotes the roots of the second factor. For a parabolic subgroup P_i of G , there corresponds to a subset J_i of the set of simple roots Π . We put $\mathbb{J} = J_1^{(1)} \sqcup J_2^{(2)}$. Here $J_i^{(i)}$ is a copy of J_i in $\Delta^{(i)}$ for $i = 1, 2$.

Let us specialize what we have already proved to the case of triple flag variety. In order to do this we will give a list of notations, which tells the correspondence of concepts.

$$\begin{aligned}
\mathbb{P} \backslash \mathbb{G} / \mathbb{P}' &= (P_1 \backslash G / P_3) \times (P_2 \backslash G / P_3) \\
&\simeq \mathbb{J} \mathbb{W} \mathbb{J}' = {}^{J_1} W^{J_3} \times {}^{J_2} W^{J_3} \ni \mathbf{w} = (w_1, w_2), \\
\mathbb{L}' &= L_3 \times L_3 \supset \mathbb{L}'_{\mathbb{K}} = \text{diag}(L_3), \\
\mathbb{U}' &= U_3 \times U_3, \\
\mathbb{P}^{\mathbf{w}} \cap \mathbb{P}' &= \mathbf{w}^{-1} \mathbb{P} \mathbf{w} \cap \mathbb{P}' = (w_1^{-1} P_1 w_1 \cap P_3) \times (w_2^{-1} P_2 w_2 \cap P_3), \\
\mathbb{P}^{\mathbf{w}} \cap \mathbb{L}' &= \mathbf{w}^{-1} \mathbb{P} \mathbf{w} \cap \mathbb{L}' = (w_1^{-1} P_1 w_1 \cap L_3) \times (w_2^{-1} P_2 w_2 \cap L_3), \\
(\mathbb{P}^{\mathbf{w}} \cap \mathbb{L}') \backslash \mathbb{L}' / \mathbb{L}'_{\mathbb{K}} &= ((P_1^{w_1} \cap L_3) \times (P_2^{w_2} \cap L_3)) \backslash (L_3 \times L_3) / \text{diag}(L_3) \\
&\stackrel{(\star)}{\simeq} (P_1^{w_1} \cap L_3) \backslash L_3 / (P_2^{w_2} \cap L_3) = \mathcal{V}(\mathbf{w}) \ni v.
\end{aligned}$$

In the case of triple flag variety, the quotient of smaller symmetric space $(\mathbb{P}^{\mathbf{w}} \cap \mathbb{L}') \backslash \mathbb{L}' / \mathbb{L}'_{\mathbb{K}}$ is just a usual Bruhat decomposition. So we can take a representative v of $\mathcal{V}(\mathbf{w})$ from the Weyl group W_{J_3} , and we further identify it with an element in $N_T(L_3) \subset L_3$. Note that, in the previous sections, $\mathcal{V}(\mathbf{w})$ is considered as a subset of $L_3 \times L_3$. Through the bijection (\star) above, we get a correspondence between them.

$$((P_1^{w_1} \cap L_3) \times (P_2^{w_2} \cap L_3)) \cdot (v, e) \cdot \text{diag}(L_3) \longleftrightarrow (P_1^{w_1} \cap L_3) \cdot v \cdot (P_2^{w_2} \cap L_3) \quad (v \in W_{J_3}).$$

Put $\mathbf{v} = (v, e) \in L_3 \times L_3$.

Let us continue reinterpreting the notations:

$$\begin{aligned}
\mathbb{P}^{\mathbf{w}\mathbf{v}} \cap \mathbb{U}' &= \mathbf{v}^{-1} \mathbf{w}^{-1} \mathbb{P} \mathbf{w} \mathbf{v} \cap \mathbb{U}' = (v^{-1} w_1^{-1} P_1 w_1 v \cap U_3) \times (w_2^{-1} P_2 w_2 \cap U_3), \\
\mathbb{P}^{\mathbf{w}\mathbf{v}} \cap \mathbb{L}'_{\mathbb{K}} &= \mathbf{v}^{-1} (\mathbb{P}^{\mathbf{w}} \cap \mathbb{L}') \mathbf{v} \cap \mathbb{L}'_{\mathbb{K}}, \\
&= \text{diag}(L_3) \cap ((v^{-1} w_1^{-1} P_1 w_1 v \cap L_3) \times (w_2^{-1} P_2 w_2 \cap L_3)), \\
&= \text{diag}(P_1^{w_1 v} \cap P_2^{w_2} \cap L_3) \supset \text{diag}(T).
\end{aligned}$$

We therefore get

$$\begin{aligned}
(\mathbb{P}^{\mathbf{w}\mathbf{v}} \cap \mathbb{U}') \backslash \mathbb{U}' / \mathbb{U}'_{\mathbb{K}} &= ((P_1^{w_1 v} \cap U_3) \times (P_2^{w_2} \cap U_3)) \backslash (U_3 \times U_3) / \text{diag}(U_3) \\
&\simeq (P_1^{w_1 v} \cap U_3) \backslash U_3 / (P_2^{w_2} \cap U_3).
\end{aligned}$$

On this last double coset space, the group $P_1^{w_1 v} \cap P_2^{w_2} \cap L_3 \simeq \mathbb{P}^{\mathbf{w}\mathbf{v}} \cap \mathbb{L}'_{\mathbb{K}}$ acts by conjugation.

Theorem 6.1. *Let P_1, P_2, P_3 be three parabolic subgroups of G . Then the diagonal G -orbits on the triple flag variety can be described as*

$$G \backslash (G/P_1 \times G/P_2 \times G/P_3) \simeq \coprod_{\tilde{P}_1, \tilde{P}_2} ((\tilde{P}_1 \cap U_3) \backslash U_3 / (\tilde{P}_2 \cap U_3)) / (\tilde{P}_1 \cap \tilde{P}_2 \cap L_3),$$

where $(\tilde{P}_1, \tilde{P}_2, P_3)$ runs over all the triples $(P_1^{w_1 v}, P_2^{w_2}, P_3)$ for $w_1 \in {}^{J_1}W^{J_3}$, $w_2 \in {}^{J_2}W^{J_3}$, and $v \in \mathcal{V}((w_1, w_2))$.

Let us consider the special case where $P_3 = B$ is a Borel subgroup. In this case $\tilde{P}_1 \cap \tilde{P}_2 \cap L_3 = T$ is a maximal torus of G and we have

$$G \setminus (G/P_1 \times G/P_2 \times G/B) \simeq \coprod_{\tilde{P}_1, \tilde{P}_2} ((\tilde{P}_1 \cap U_B) \backslash U_B / (\tilde{P}_2 \cap U_B)) / T.$$

Also by applying Theorem 4.5 to the present case, we conclude that:

Corollary 6.2. *A triple flag variety $G/P_1 \times G/P_2 \times G/B$ is of finite type if and only if $\mathfrak{u}_{J_1} \cap \mathfrak{u}_{J_2}$ is $L_{J_1 \cap J_2}$ -spherical.*

The triple flag varieties $G/P_1 \times G/P_2 \times G/B$ of finite type (or equivalently the G -spherical double flag varieties $G/P_1 \times G/P_2$) were classified by Stembridge [Ste03] in connection with multiplicity-free tensor product of two irreducible G -modules.

Theorem 6.3 ([Ste03]). *Let G be a connected simple algebraic group. Let P_1, P_2 be parabolic subgroups of G corresponding to sets of simple roots $J_1, J_2 \subsetneq \Pi$, respectively. Then the triple flag variety $G/P_1 \times G/P_2 \times G/B$ is of finite type if and only if the pair $\langle \Pi \setminus J_1, \Pi \setminus J_2 \rangle$ appears in Table 1 up to switching J_1 and J_2 .*

If we assume further that P_2 is a Borel subgroup, we see from Table 1 that:

Corollary 6.4. *Let G be a connected simple algebraic group and P a parabolic subgroup. Then the triple flag variety $G/P \times G/B \times G/B$ is of finite type if and only if $G = SL_n$ and G/P is isomorphic to the projective space of dimension $n - 1$ (i.e. P is a mirabolic subgroup of G).*

Remark 6.5. Recently Tanaka [Tan12] proved that $G/P_1 \times G/P_2$ is G -spherical if and only if the action of a compact real form of G on $G/P_1 \times G/P_2$ is strongly visible.

\mathfrak{g}	$\langle \Pi \setminus J_1, \Pi \setminus J_2 \rangle$ up to switching J_1 and J_2
\mathfrak{sl}_{n+1}	$\begin{array}{c} \alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_n \\ \circ \text{---} \circ \text{---} \dots \text{---} \circ \\ \langle \{\alpha_i\}, \{\alpha_j\} \rangle (\forall i, j), \langle \{\alpha_1, \alpha_i\}, \{\alpha_j\} \rangle (\forall i, j), \\ \langle \{\alpha_i, \alpha_n\}, \{\alpha_j\} \rangle (\forall i, j), \langle \{\alpha_i, \alpha_{i+1}\}, \{\alpha_j\} \rangle (\forall i, j), \\ \langle \{\alpha_i, \alpha_j\}, \{\alpha_2\} \rangle (\forall i, j), \langle \{\alpha_i, \alpha_j\}, \{\alpha_{n-1}\} \rangle (\forall i, j), \\ \langle \{\alpha_1\}, \text{any} \rangle, \langle \{\alpha_n\}, \text{any} \rangle \end{array}$
\mathfrak{so}_{2n+1}	$\begin{array}{c} \alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_{n-1} \quad \alpha_n \\ \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \\ \langle \{\alpha_1\}, \{\alpha_i\} \rangle (\forall i), \langle \{\alpha_n\}, \{\alpha_n\} \rangle \end{array}$

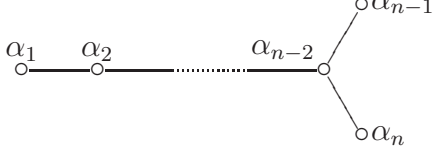
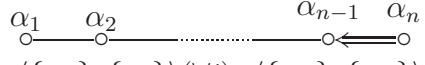
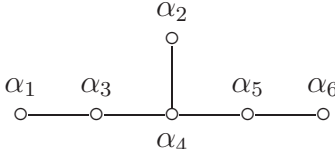
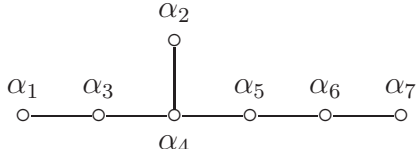
\mathfrak{so}_{2n}	
$n \geq 4$	$\begin{aligned} &\langle \{\alpha_1\}, \{\alpha_i\} \rangle (\forall i), \langle \{\alpha_i\}, \{\alpha_j\} \rangle (i = 1, 2, 3, j = n-1, n), \\ &\langle \{\alpha_{n-1}\}, \{\alpha_{n-1}\} \rangle, \langle \{\alpha_{n-1}\}, \{\alpha_n\} \rangle, \langle \{\alpha_n\}, \{\alpha_n\} \rangle, \\ &\langle \{\alpha_1\}, \{\alpha_i, \alpha_{n-1}\} \rangle (\forall i), \langle \{\alpha_1\}, \{\alpha_i, \alpha_n\} \rangle (\forall i), \\ &\langle \{\alpha_{n-1}\}, \{\alpha_1, \alpha_2\} \rangle, \langle \{\alpha_n\}, \{\alpha_1, \alpha_2\} \rangle, \\ &\langle \{\alpha_i\}, \{\alpha_j, \alpha_k\} \rangle (i = n-1, n, \{j, k\} \subset \{1, n-1, n\}), \\ &(\langle \{\alpha_3\}, \{\alpha_2, \alpha_4\} \rangle, \langle \{\alpha_4\}, \{\alpha_2, \alpha_3\} \rangle \text{ if } n = 4) \end{aligned}$
\mathfrak{sp}_n	
	$\langle \{\alpha_1\}, \{\alpha_i\} \rangle (\forall i), \langle \{\alpha_n\}, \{\alpha_n\} \rangle$
\mathfrak{e}_6	
	$\langle \{\alpha_i\}, \{\alpha_j\} \rangle (i = 1, 6, j \neq 4), \quad \langle \{\alpha_1\}, \{\alpha_1, \alpha_6\} \rangle, \langle \{\alpha_6\}, \{\alpha_1, \alpha_6\} \rangle$
\mathfrak{e}_7	
	$\langle \{\alpha_7\}, \{\alpha_1\} \rangle, \langle \{\alpha_7\}, \{\alpha_2\} \rangle, \langle \{\alpha_7\}, \{\alpha_7\} \rangle$

Table 1: $G/P_1 \times G/P_2 \times G/B$ of finite type7. CLASSIFICATION OF K -SPHERICAL G/P

In this section, we give a classification of the triples (G, K, P) such that $G/P \times K/B_K$ is of finite type, where B_K is a Borel subgroup of K . It is known that any symmetric pair (\mathbb{G}, \mathbb{K}) with \mathbb{G} connected, simply connected, and semisimple is a direct product of symmetric pairs (G, K) such that

- G is simple, or
- $G = G' \times G'$, $K = \text{diag } G'$, and G' is simple.

We note that there is a symmetric pair $(G' \times G', G'_s)$, where s is an automorphism of G' and $G'_s = \{(g, s(g)) : g \in G'\}$, but it is isomorphic to $(G' \times G', \text{diag}(G'))$ by the map $G' \times G' \rightarrow G' \times G'$, $(g_1, g_2) \mapsto (g_1, s(g_2))$. Thus it is enough to treat the two cases above. For the latter case, $G/P \times K/B_K$ can be written as the triple flag variety $G'/P'_1 \times G'/P'_2 \times G'/B_{G'}$ and the classification was already given (see Theorem 6.1 and [Ste03]).

In the rest of this section we assume that G is simple.

We first consider the case where (G, K) is the complexification of a Hermitian symmetric pair, or equivalently the center of K is one-dimensional. In this case, K equals the Levi component of a maximal parabolic subgroup of G . Therefore we can choose a θ -stable Borel subgroup B of G and a simple root $\alpha_i \in \Pi$ such that $K = L_{\Pi \setminus \{\alpha_i\}}$. Then $K = P_{\Pi \setminus \{\alpha_i\}} \cap P_{\Pi \setminus \{\alpha_i\}}^-$, where $P_{\Pi \setminus \{\alpha_i\}}^-$ is the opposite parabolic subgroup of $P_{\Pi \setminus \{\alpha_i\}}$.

Lemma 7.1. *Suppose that (G, K) is the complexification of a Hermitian symmetric pair and P is a parabolic subgroup of G . Choose a θ -stable Borel subgroup B and a simple root α_i such that $K = L_{\Pi \setminus \{\alpha_i\}}$. Then $G/P \times K/B_K$ is of finite type if and only if $G/P \times G/P_{\Pi \setminus \{\alpha_i\}}^* \times G/B$ is of finite type. Here $\Pi \setminus \{\alpha_i\}^* := -w_0(\Pi \setminus \{\alpha_i\})$ for the longest element $w_0 \in W$.*

Proof. The opposite parabolic subgroup $P_{\Pi \setminus \{\alpha_i\}}^-$ of $P_{\Pi \setminus \{\alpha_i\}}$ is conjugate to $P_{\Pi \setminus \{\alpha_i\}}^*$. Hence $G/P_{\Pi \setminus \{\alpha_i\}}^- \simeq G/P_{\Pi \setminus \{\alpha_i\}}^*$. We have

$$P_{\Pi \setminus \{\alpha_i\}}^- \cap B = P_{\Pi \setminus \{\alpha_i\}}^- \cap P_{\Pi \setminus \{\alpha_i\}} \cap B = K \cap B = B_K$$

and $P_{\Pi \setminus \{\alpha_i\}}^- \cdot B$ is open in G . Using an argument of [NO11, Theorem 2], we see that $G/P \times K/B_K$ is of finite type if and only if $G/P \times G/P_{\Pi \setminus \{\alpha_i\}}^* \times G/B$ is of finite type. \square

By Theorem 6.3 and Lemma 7.1, we get a list of $G/P \times K/B_K$ of finite type.

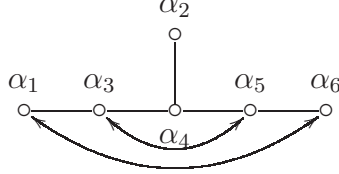
For the remaining pairs (G, K) , we use Theorem 4.6 and the classification of spherical linear actions by Benson and Ratcliff [BR96]. We carry out a classification according to the following procedure.

- (1) For each symmetric pair (G, K) we fix a θ -stable Borel subgroup of B .
- (2) Take a standard parabolic subgroup P and determine $\mathfrak{l} \cap \mathfrak{k}$ and $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$.
- (3) Check whether the $(L \cap K)$ -action on $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$ is spherical using the list of [BR96].

In addition, the obvious dimension condition $\dim G/P + \dim K/B_K \leq \dim K$ is helpful in some cases. We note that the choice of a θ -stable Borel subgroup of B is not unique in general and the Lie algebras $\mathfrak{l} \cap \mathfrak{k}$ and $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$ depend on this choice. For our purpose, it is enough to check Theorem 4.6 (3) for one choice of B .

We give a computation for one example.

Example 7.2. Let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{e}_6, \mathfrak{f}_4)$. We fix a θ -stable Borel subgroup B corresponding to the following Vogan diagram.



This means that if \mathfrak{g}_α denotes the root space corresponding to $\alpha \in \Delta$, then we have $\mathfrak{g}_{\alpha_2}, \mathfrak{g}_{\alpha_4} \subset \mathfrak{k}$, $\theta \mathfrak{g}_{\alpha_1} = \mathfrak{g}_{\alpha_6}$, and $\theta \mathfrak{g}_{\alpha_3} = \mathfrak{g}_{\alpha_5}$.

We now see the dimension condition. Since $\dim K/B_K = 24$ and $\dim K = 52$, we must have $\dim G/P_J \leq 52 - 24 = 28$ if $G/P_J \times K/B_K$ is of finite type. This inequality is satisfied for $\Pi \setminus J = \{\alpha_i\} (i \neq 4)$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_2, \alpha_6\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_5, \alpha_6\}$, $\{\alpha_1, \alpha_6\}$.

Let $\Pi \setminus J = \{\alpha_1, \alpha_2\}$. Then we have $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{sp}_2 \oplus \mathbb{C}^2$ and $\mathfrak{u} \cap \mathfrak{g}^{-\theta} = V_1 \oplus V_2 = \mathbb{C}^4 \oplus \mathbb{C}^2$. Here the action is given in the following way: \mathfrak{sp}_2 acts naturally on V_1 and trivially on V_2 ; the center \mathbb{C}^2 of $\mathfrak{l} \cap \mathfrak{k}$ acts faithfully on V_2 . Since V_1 is Sp_2 -spherical, $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$ is $(L \cap K)$ -spherical.

Let $\Pi \setminus J = \{\alpha_1, \alpha_3\}$. Then we have $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{sl}_3 \oplus \mathbb{C}^2$ and $\mathfrak{u} \cap \mathfrak{g}^{-\theta} = V_1 \oplus V_2 = \mathbb{C}^3 \oplus \mathbb{C}^2$. Here the action is given in the following way: \mathfrak{sl}_3 acts naturally on V_1 and trivially on V_2 ; the center \mathbb{C}^2 of $\mathfrak{l} \cap \mathfrak{k}$ acts faithfully on V_2 . Since V_1 is SL_3 -spherical, $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$ is $(L \cap K)$ -spherical.

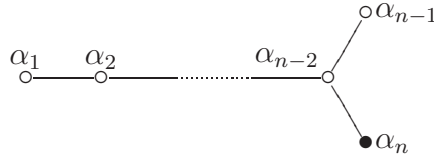
Let $\Pi \setminus J = \{\alpha_1, \alpha_6\}$. Then we have $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{so}_7 \oplus \mathbb{C}$ and $\mathfrak{u} \cap \mathfrak{g}^{-\theta} = V_1 \oplus V_2 = \mathbb{C}^8 \oplus \mathbb{C}$. Here the action is given in the following way: \mathfrak{so}_7 acts on V_1 as a spin representation and acts trivially on V_2 ; the center \mathbb{C} of $\mathfrak{l} \cap \mathfrak{k}$ acts faithfully on V_2 . According to [BR96], V_1 is not $Spin_7$ -spherical and hence $\mathfrak{u} \cap \mathfrak{g}^{-\theta}$ is not $(L \cap K)$ -spherical.

From the argument above, we can see that $G/P_J \times K/B_K$ is of finite type if and only if $\Pi \setminus J = \{\alpha_i\} (i \neq 4)$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_2, \alpha_6\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_5, \alpha_6\}$.

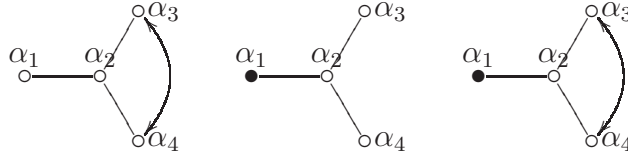
We can compute in a similar way for other symmetric pairs and obtain a classification of $G/P \times K/B_K$ of finite type.

Theorem 7.3. *Let G be a connected simple algebraic group and (G, K) a symmetric pair. Let P be a parabolic subgroup of G corresponding to $J \subsetneq \Pi$. Then the double flag variety $G/P \times K/B_K$ is of finite type if and only if the triple $(\mathfrak{g}, \mathfrak{k}, \Pi \setminus J)$ appears in Table 2.*

Remark 7.4. For $\mathfrak{g} \simeq \mathfrak{so}_{4n}$, a symmetric subalgebra \mathfrak{k} that is isomorphic to $\mathfrak{sl}_{2n} \oplus \mathbb{C}$ is not unique up to inner automorphisms of \mathfrak{g} . For $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}_{4n}, \mathfrak{sl}_{2n} \oplus \mathbb{C})$ in Table 2, we take $(\mathfrak{g}, \mathfrak{k})$ and a positive system Δ^+ in such a way that the Vogan diagram becomes



Similarly, the subalgebra \mathfrak{k} of \mathfrak{g} is not unique for $(\mathfrak{g}, \mathfrak{k}) \simeq (\mathfrak{so}_8, \mathfrak{so}_7)$, $(\mathfrak{so}_8, \mathfrak{so}_6 \oplus \mathbb{C})$, $(\mathfrak{so}_8, \mathfrak{so}_5 \oplus \mathfrak{so}_3)$. In Table 2, we take $(\mathfrak{g}, \mathfrak{k})$ and positive systems Δ^+ in such a way that the Vogan diagrams become



\mathfrak{g}	\mathfrak{k}	$\Pi \setminus J \ (P = P_J)$
\mathfrak{sl}_{n+1}		$\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_n$
\mathfrak{sl}_{n+1}	\mathfrak{so}_{n+1}	$\{\alpha_i\}(\forall i)$
\mathfrak{sl}_{2m} $2m = n + 1$	\mathfrak{sp}_m	$\{\alpha_i\}(\forall i), \{\alpha_i, \alpha_{i+1}\}(\forall i),$ $\{\alpha_1, \alpha_i\}(\forall i), \{\alpha_i, \alpha_n\}(\forall i),$ $\{\alpha_1, \alpha_2, \alpha_3\}, \{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\},$ $\{\alpha_1, \alpha_2, \alpha_n\}, \{\alpha_1, \alpha_{n-1}, \alpha_n\}$
\mathfrak{sl}_{p+q} $p + q = n + 1$	$\mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{C}$ $p \geq q \geq 1$	$\{\alpha_i\}(\forall i), \{\alpha_i, \alpha_{i+1}\}(\forall i),$ $\{\alpha_1, \alpha_i\}(\forall i), \{\alpha_i, \alpha_n\}(\forall i),$ $(\{\alpha_i, \alpha_j\}(\forall i, j) \text{ if } q = 2),$ $(\text{any subset of } \Pi \text{ if } q = 1)$
\mathfrak{so}_{2n+1}		$\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_{n-1} \quad \alpha_n$
\mathfrak{so}_{p+q} $p + q = 2n + 1$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$ $p \geq q \geq 1$	$\{\alpha_1\}, \{\alpha_n\},$ $(\{\alpha_i\}(\forall i) \text{ if } q = 2),$ $(\text{any subset of } \Pi \text{ if } q = 1)$
\mathfrak{so}_{2n}		
\mathfrak{so}_{p+q} $p + q = 2n$ $n \geq 4$	${}^1\mathfrak{so}_p \oplus \mathfrak{so}_q$ $p \geq q \geq 1$	$\{\alpha_1\}, \{\alpha_{n-1}\}, \{\alpha_n\},$ $(\{\alpha_i\}(\forall i) \text{ if } q = 2),$ $(\{\alpha_i, \alpha_{n-1}\}(\forall i) \text{ if } q = 2),$ $(\{\alpha_i, \alpha_n\}(\forall i) \text{ if } q = 2),$

¹See Remark 7.4

		(any subset of Π if $q = 1$)
\mathfrak{so}_{2n} $n \geq 4$	${}^1\mathfrak{sl}_n \oplus \mathbb{C}$	$\{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_{n-1}\}, \{\alpha_n\},$ $\{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_{n-1}\}, \{\alpha_1, \alpha_n\}, \{\alpha_{n-1}, \alpha_n\},$ $(\{\alpha_2, \alpha_3\} \text{ if } n = 4)$
\mathfrak{sp}_n		$\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n-1} \quad \alpha_n$
\mathfrak{sp}_n	$\mathfrak{sl}_n \oplus \mathbb{C}$	$\{\alpha_1\}, \{\alpha_n\}$
\mathfrak{sp}_{p+q} $p+q = n$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$ $p \geq q$	$\{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_n\}, \{\alpha_1, \alpha_2\},$ $(\{\alpha_i\}(\forall i) \text{ if } q \leq 2),$ $(\{\alpha_i, \alpha_j\}(\forall i, j) \text{ if } q = 1)$
\mathfrak{f}_4		$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$
\mathfrak{f}_4	\mathfrak{so}_9	$\{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_4\}, \{\alpha_1, \alpha_4\}$
\mathfrak{e}_6		α_2
\mathfrak{e}_6	\mathfrak{sp}_4	$\{\alpha_1\}, \{\alpha_6\}$
\mathfrak{e}_6	$\mathfrak{sl}_6 \oplus \mathfrak{sl}_2$	$\{\alpha_1\}, \{\alpha_6\}$
\mathfrak{e}_6	$\mathfrak{so}_{10} \oplus \mathbb{C}$	$\{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_5\}, \{\alpha_6\}, \{\alpha_1, \alpha_6\}$
\mathfrak{e}_6	\mathfrak{f}_4	$\{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_5\}, \{\alpha_6\},$ $\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_6\}, \{\alpha_1, \alpha_3\}, \{\alpha_5, \alpha_6\}$
\mathfrak{e}_7		α_2
\mathfrak{e}_7	\mathfrak{sl}_8	$\{\alpha_7\}$
\mathfrak{e}_7	$\mathfrak{so}_{12} \oplus \mathfrak{sl}_2$	$\{\alpha_7\}$
\mathfrak{e}_7	$\mathfrak{e}_6 \oplus \mathbb{C}$	$\{\alpha_1\}, \{\alpha_2\}, \{\alpha_7\}$

Table 2: K -spherical G/P

8. CLASSIFICATION OF G -SPHERICAL G/Q

In this section, we give a classification of the triples (G, K, Q) such that $G/B \times K/Q$ is of finite type. As in the previous section, it is enough to consider the two cases:

- G is simple;
- $G = G' \times G'$, $K = \text{diag } G'$, and G' is simple.

In the latter case, $G/B \times K/Q$ can be written as the triple flag variety $G'/B_{G'} \times G'/B_{G'} \times G'/Q$ and the classification was already given (see Corollary 6.4).

In what follows we assume that G is simple.

We first consider the case where (G, K) is the complexification of a Hermitian symmetric pair. We choose a θ -stable Borel subgroup B of G and a simple root $\beta \in \Pi$ such that $K = L_{\Pi \setminus \{\beta\}}$. Since $\text{rank } G = \text{rank } K$, we have $T = T_K$. Therefore the set of simple roots Π_K for K can be regarded as a subset of Π and then $\Pi = \Pi_K \cup \{\beta\}$.

Lemma 8.1. *Suppose that (G, K) is the complexification of a Hermitian symmetric pair and Q_{J_K} is the parabolic subgroup of K corresponding to a subset $J_K \subset \Pi_K (\subset \Pi)$. Choose a θ -stable Borel subgroup B and a simple root β such that $K = L_{\Pi \setminus \{\beta\}}$. Then $G/B \times K/Q_{J_K}$ is of finite type if and only if $G/B \times G/P_{J_K} \times G/P_{\Pi \setminus \{\beta\}^*}$ is of finite type.*

Proof. The opposite parabolic subgroup $P_{\Pi \setminus \{\beta\}}^-$ is conjugate to $P_{\Pi \setminus \{\beta\}^*}$ and hence $G/P_{\Pi \setminus \{\beta\}}^- \simeq G/P_{\Pi \setminus \{\beta\}^*}$. We have

$$P_{\Pi \setminus \{\beta\}}^- \cap P_{J_K} = P_{\Pi \setminus \{\beta\}}^- \cap P_{\Pi \setminus \{\beta\}} \cap P_{J_K} = K \cap P_{J_K} = Q_{J_K}$$

and $P_{\Pi \setminus \{\beta\}}^- \cdot P_{J_K}$ is open in G . Hence the lemma follows from [NO11, Theorem 2] (see also [KNOT12]). \square

By Theorem 6.3 and Lemma 8.1, we get a list of $G/B \times K/Q$ of finite type.

For the remaining pairs (G, K) , we use Theorem 5.2 and the classification of spherical linear actions in [BR96]. We carry out a classification according to the following procedure.

- (1) For each triple (G, K, Q) we choose a θ -stable parabolic subgroup of P' such that $P' \cap K = Q$.
- (2) Determine the Lie algebras \mathfrak{l}' , $\mathfrak{l}' \cap \mathfrak{k}$ and $(\mathfrak{u}')^{-\theta}$.
- (3) Determine \mathfrak{m}' (see Theorem 5.2) and check whether the M'_0 -action on $(\mathfrak{u}')^{-\theta}$ is spherical using the list of [BR96].

The dimension condition $\dim G/B + \dim K/Q \leq \dim K$ is also helpful in some cases.

Example 8.2. Let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{e}_6, \mathfrak{f}_4)$. We label the simple roots for K and for G as in the following diagrams.



Since $\dim G/B = 36$ and $\dim K = 52$, we must have $\dim K/Q \leq 52 - 36 = 16$ if $G/B \times K/Q$ is of finite type. This inequality is satisfied for $\Pi_K \setminus J_K = \{\alpha_1\}, \{\alpha_4\}$.

Let $\Pi_K \setminus J_K = \{\alpha_1\}$. Then we can take P' as $P' = P_{\Pi \setminus \{\beta_2\}}$. We have $\mathfrak{l}' = \mathfrak{sl}_6 \oplus \mathbb{C}$, $\mathfrak{l}' \cap \mathfrak{k} = \mathfrak{sp}_3 \oplus \mathbb{C}$, $\mathfrak{m}' = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}$, and $(\mathfrak{u}')^{-\theta} = \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}$. Here the action is given in the following way: each factor \mathfrak{sl}_2 acts naturally on one component \mathbb{C}^2 and acts trivially on the other components; the center \mathbb{C} of $\mathfrak{l}' \cap \mathfrak{k}$ acts faithfully on \mathbb{C} . Since \mathbb{C}^2 is SL_2 -spherical, $(\mathfrak{u}')^{-\theta}$ is M'_0 -spherical.

Let $\Pi_K \setminus J_K = \{\alpha_4\}$. Then we can take P' as $P' = P_{\Pi \setminus \{\beta_1, \beta_6\}}$. Since $G/P' \times K/B_K$ is not of finite type, $G/B \times K/Q_{J_K}$ is not of finite type by Corollary 5.3.

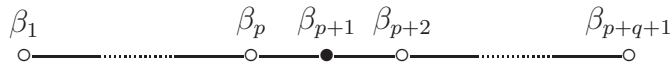
Therefore, $G/B \times K/Q_{J_K}$ is of finite type if and only if $\Pi_K \setminus J_K = \{\alpha_1\}$.

We can compute in a similar way for other symmetric pairs and obtain a classification of $G/B \times K/Q$ of finite type.

Theorem 8.3. *Let G be a connected simple algebraic group and (G, K) a symmetric pair. Let Q be a parabolic subgroup of K corresponding to $J_K \subsetneq \Pi_K$. Then the double flag variety $G/B \times K/Q$ is of finite type if and only if the triple $(\mathfrak{g}, \mathfrak{k}, \Pi_K \setminus J_K)$ appears in Table 3.*

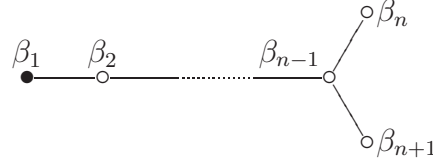
Remark 8.4. For some pairs $(\mathfrak{g}, \mathfrak{k})$ in Table 3, the labeling $\alpha_1, \dots, \alpha_n$ of the simple roots for \mathfrak{k} is not unique. In order to fix this, we take a θ -stable Borel subgroup B' of G containing T and give a label $\alpha_1, \dots, \alpha_n$ in terms of the Vogan diagram for (G, B', K) . We remark that P' is not necessarily standard with respect to B' .

For $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_{p+q+2}, \mathfrak{sl}_{p+1} \oplus \mathfrak{sl}_{q+1} \oplus \mathbb{C})$ in Table 3, we can take B' such that the Vogan diagram for (G, B', K) becomes



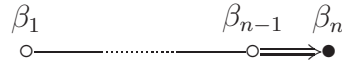
We put $\alpha_i := \beta_i$ for $1 \leq i \leq p$ and $\alpha_{p+i} := \beta_{p+i+1}$ for $1 \leq i \leq q$ so that $\alpha_1, \dots, \alpha_{p+q}$ form a set of simple roots for K .

For $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}_{2n+2}, \mathfrak{so}_{2n} \oplus \mathbb{C})$ in Table 3, we can take B' such that the Vogan diagram for (G, B', K) becomes



We put $\alpha_i := \beta_{i+1}$ for $1 \leq i \leq n$ so that $\alpha_1, \dots, \alpha_n$ form a set of simple roots for K .

For $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})$ in Table 3, we can take B' such that the Vogan diagram for (G, B', K) becomes



We put $\alpha_i := \beta_i$ for $1 \leq i \leq n-1$ and $\alpha_n := \beta_{n-1} + 2\beta_n$ so that $\alpha_1, \dots, \alpha_n$ form a set of simple roots for K .

Remark 8.5. In Table 3, we also describe the K -module $\mathfrak{g}^{-\theta}$. We write $\omega_i \in \mathfrak{t}_K^*$ for the fundamental weight corresponding to α_i and write $V(\lambda)$ for the irreducible K -module with highest weight λ .

\mathfrak{g}	\mathfrak{k}	$\mathfrak{g}^{-\theta}$ (See Remark 8.5)	$\Pi_K \setminus J_K$ ($Q = Q_{J_K}$)
\mathfrak{sl}_{2n} $n \geq 2$	\mathfrak{sp}_n	$V(\omega_2)$	 $\{\alpha_1\},$ $(\{\alpha_3\} \text{ if } n = 3),$ $(\text{any subset of } \Pi_K \text{ if } n = 2)$
\mathfrak{sl}_{p+q+2} $p+q \geq 1$	$\mathfrak{sl}_{p+1} \oplus \mathfrak{sl}_{q+1} \oplus \mathbb{C}$ $p \geq q$	$V(\omega_1 + \omega_{p+q}) \oplus V(\omega_p + \omega_{p+1})$ $(V(\omega_1) \oplus V(\omega_p) \text{ if } q = 0)$	 $\{\alpha_1\}, \{\alpha_p\}, \{\alpha_{p+1}\}, \{\alpha_{p+q}\},$ $(\{\alpha_i\}(\forall i) \text{ if } q = 1),$ $(\text{any subset of } \Pi_K \text{ if } q = 0)$
\mathfrak{so}_{2n+2} $n \geq 3$	$\mathfrak{so}_{2n} \oplus \mathbb{C}$	$V(\omega_1) \oplus V(\omega_1)$	 $\{\alpha_{n-1}\}, \{\alpha_n\}$

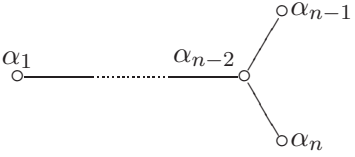
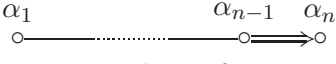
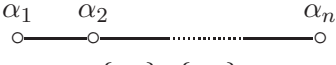
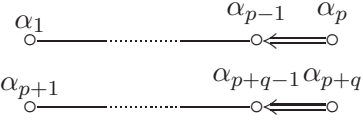
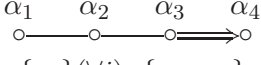
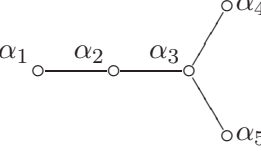
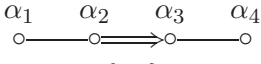
\mathfrak{so}_{2n+1} $n \geq 3$	\mathfrak{so}_{2n}	$V(\omega_1)$	 any subset of Π_K
\mathfrak{so}_{2n+2} $n \geq 3$	\mathfrak{so}_{2n+1}	$V(\omega_1)$	 any subset of Π_K
\mathfrak{so}_{2n+2} $n \geq 3$	$\mathfrak{sl}_{n+1} \oplus \mathbb{C}$	$V(\omega_2) \oplus V(\omega_{n-1})$	 $\{\alpha_1\}, \{\alpha_n\}$
\mathfrak{sp}_{p+q}	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$ $p \geq q \geq 1$	$V(\omega_1 + \omega_{p+1})$	 $\{\alpha_1\}, \{\alpha_{p+1}\},$ $(\{\alpha_{p+q}\} \text{ if } q \leq 3),$ $(\{\alpha_p\} \text{ if } p \leq 3),$ $(\{\alpha_p\} \text{ if } q \leq 2),$ $(\{\alpha_1, \alpha_2\} \text{ if } p = 2),$ $(\{\alpha_{p+1}, \alpha_{p+2}\} \text{ if } q = 2),$ $(\{\alpha_i\}(\forall i), \{\alpha_i, \alpha_j\}(\forall i, j) \text{ if } q = 1)$
\mathfrak{f}_4	\mathfrak{so}_9	$V(\omega_4)$	 $\{\alpha_i\}(\forall i), \{\alpha_1, \alpha_2\}$
\mathfrak{e}_6	$\mathfrak{so}_{10} \oplus \mathbb{C}$	$V(\omega_4) \oplus V(\omega_5)$	 $\{\alpha_1\}$
\mathfrak{e}_6	\mathfrak{f}_4	$V(\omega_4)$	 $\{\alpha_1\}$

Table 3: G -spherical G/Q

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